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Motion of a graph by R -curvature (Viscosity Solutions of Differential Equations and Related Topics)

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Motion of a graph by R -curvature

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1. Introduction.

In this talk we introduce our recent result:

H. Ishii and T. Mikami, Motion of a graph by R -curvature, Hokkaido mathematical preprint series, No. 340.

Let us first introduce two definitions.

Definition 1 (R -curvature) Let $R \in L^1(\mathbf{R}^d : [0, \infty), dx)$. For $u \in C(\mathbf{R}^d : \mathbf{R})$, we define the R -curvature of u as the finite Borel measure $w(R, u, dx)$ on \mathbf{R}^d given by

$$w(R, u, A) \equiv \int_{\bigcup_{x \in A} \partial u(x)} R(y) dy \quad \text{for all Borel } A \subset \mathbf{R}^d. \quad (0.1)$$

Definition 2 (Motion by R -curvature) The graph of $u \in C([0, \infty) \times \mathbf{R}^d : \mathbf{R})$ is called the motion by R -curvature if the following holds: for any $\varphi \in C_o(\mathbf{R}^d : \mathbf{R})$ and any $t \geq 0$,

$$\begin{aligned} & \int_{\mathbf{R}^d} \varphi(x) u(t, x) dx - \int_{\mathbf{R}^d} \varphi(x) u(0, x) dx \\ &= \int_0^t ds \int_{\mathbf{R}^d} \varphi(x) w(R, u(s, \cdot), dx). \end{aligned} \quad (0.2)$$

By the continuum limit of a class of infinite particle systems, we first show the existence of the motion by R -curvature, and then the uniqueness by the comparison theorem. We also show that the motion by R -curvature is a viscosity solution to

$$(PDE) \quad \partial u(t, x) / \partial t = \chi(u, Du(t, x), t, x) \text{Det}_+(D^2 u(t, x)) R(Du(t, x)),$$

where $Du(t, x) \equiv (\partial u(t, x) / \partial x_i)_{i=1}^d$, $D^2 u(t, x) \equiv (\partial^2 u(t, x) / \partial x_i \partial x_j)_{i,j=1}^d$,

$$\chi(u, p, t, x) \equiv \begin{cases} 1 & \text{if } p \in \partial u(t, x), \\ 0 & \text{otherwise,} \end{cases}$$

$\partial u(t, x)$ denotes the subdifferential of the function $x \mapsto u(t, x)$, and for a real $d \times d$ -symmetric matrix X ,

$$\text{Det}_+ X \equiv \begin{cases} \text{Det} X & \text{if } X \text{ is nonnegative definite,} \\ 0 & \text{otherwise.} \end{cases}$$

We introduce the definition of the viscosity solution to (PDE).

Definition 3 (Viscosity solution) (*Viscosity subsolution*) $u \in C((0, \infty) \times \mathbf{R}^d : \mathbf{R})$ is a viscosity subsolution of (PDE) if whenever $\varphi \in C^2((0, \infty) \times \mathbf{R}^d : \mathbf{R})$ and $u - \varphi \leq (u - \varphi)(t_o, x_o)$,

$$\partial\varphi(t_o, x_o)/\partial t \leq \chi(u, D\varphi(t_o, x_o), t_o, x_o) \text{Det}_+(D^2\varphi(t_o, x_o))R(D\varphi(t_o, x_o)).$$

(Viscosity supersolution) $u \in C((0, \infty) \times \mathbf{R}^d : \mathbf{R})$ is a viscosity supersolution of (PDE) if whenever $\varphi \in C^2((0, \infty) \times \mathbf{R}^d : \mathbf{R})$ and $u - \varphi \geq (u - \varphi)(t_o, x_o)$,

$$\partial\varphi(t_o, x_o)/\partial t \geq \chi^-(u, D\varphi(t_o, x_o), t_o, x_o) \text{Det}_+(D^2\varphi(t_o, x_o))R(D\varphi(t_o, x_o)).$$

Here $\chi^-(v, p, t, x) = 1$ if

$$v(t, y) > v(t, x) + \langle p, y - x \rangle \quad (y \neq x)$$

and if there exists $\varepsilon > 0$ such that for all $(s, y) \in (0, \infty) \times \mathbf{R}^d$ satisfying $|y| > \varepsilon^{-1}$ and $|s - t| < \varepsilon$,

$$v(s, y) > p \cdot y + \varepsilon|y|,$$

and $\chi^-(v, p, t, x) = 0$, otherwise.

Remark 1 If $\chi^-(v, p, t, x) = 1$ and s is close to t , then $p \in \partial v(s, y)$ for some y .

Finally we discuss under what condition the viscosity solution to (PDE) is the motion by R -curvature.

2. Infinite particle systems and the motion by R -curvature.

In this section we construct the motion by R -curvature by the continuum limit of infinite particle systems.

Fix $\varepsilon_n \downarrow 0$ as $n \rightarrow \infty$, and put

$$(A.1). \quad \|R\|_{L^1} \equiv \int_{\mathbf{R}^d} R(y) dy > 0, \quad R \geq 0, \quad h \in C(\mathbf{R}^d : \mathbf{R}),$$

$$(A.2). \quad |\partial h(\mathbf{R}^d)| (\equiv \cup_{x \in \mathbf{R}^d} \partial h(x)) > 0,$$

$$S_n \equiv \{v : \mathbf{Z}^d/n \mapsto \mathbf{R} \mid \sum_{z \in \mathbf{Z}^d/n} (v(z) - h(z)) < \infty, \\ (v(z) - h(z))/\varepsilon_n \in \mathbf{N} \cup \{0\} \text{ for all } z \in \mathbf{Z}^d/n\}.$$

Let $\{Y_n(k, \cdot)\}_{0 \leq k}$ be a Markov chain on S_n such that $Y_n(0, \cdot) = h(\cdot)$, and that

$$P(Y_n(k+1, \cdot) = v_{n,z} \mid Y_n(k, \cdot) = v) = w(R, \hat{v}, \{z\})/w(R, \hat{Y}_n(0, \cdot), \mathbf{R}^d),$$

where

$$v_{n,z}(x) \equiv \begin{cases} v(x) + \varepsilon_n & \text{if } x = z, \\ v(x) & \text{if } x \in (\mathbf{Z}^d/n) \setminus \{z\}. \end{cases}$$

Let $p_n(t)$ be a Poisson process, with parameter $n^d \varepsilon_n^{-1} w(R, \hat{Y}_n(0, \cdot), \mathbf{R}^d)$, which is independent of Y_n . Put

$$Z_n(t, z) \equiv Y_n(p_n(t), z),$$

$$X_n(t, x) \equiv \max(\hat{Z}_n(t, x), h(x)).$$

For f and $g \in C(\mathbf{R}^d : \mathbf{R})$, we put

$$d_{C(\mathbf{R}^d : \mathbf{R})}(f, g) \equiv \sum_{m \geq 1} 2^{-m} \min(\sup_{|x| \leq m} |f(x) - g(x)|, 1).$$

Then we show that $X_n(t, x)$ converges to the motion by R -curvature under the following additional conditions.

(A.3). The closure of the set $\{x \in \mathbf{R}^d : \hat{h}(x) < h(x)\}$ does not contain any line which is unbounded in two different directions.

(A.4). For any $p \notin \partial h(\mathbf{R}^d)$ and $C \in \mathbf{R}$,

$$\int_{\mathbf{R}^d} \max(\langle p, x \rangle + C - h(x), 0) dx = \infty.$$

Theorem 1 *Suppose that (A.1) and (A.3)-(A.4) hold. Then there exists a unique continuous solution u to (1.2) with $u(0, \cdot) = h$. Suppose in addition that (A.2) holds. Then the following holds: for any $\gamma > 0$ and $T > 0$,*

$$\lim_{n \rightarrow \infty} P\left(\sup_{0 \leq t \leq T} d_{C(\mathbf{R}^d; \mathbf{R})}(X_n(t, \cdot), u(t, \cdot)) \geq \gamma\right) = 0.$$

Remark 2 *(A.3) holds when $d = 1$. If h is convex, then (A.4) holds.*

We give the properties of the motion by R -curvature.

Theorem 2 *Suppose that (A.1) holds. Let $u \in C([0, \infty) \times \mathbf{R}^d : \mathbf{R})$ be the solution to (1.2) with $u(0, \cdot) = h$. Then:*

(a) $t \mapsto u(t, x)$ is nondecreasing.

(b) $u = \max(\hat{u}, h)$

(c) $u(t, x) - \hat{u}(t, x) \leq h(x) - \hat{h}(x)$. In particular, if $h(x) = \hat{h}(x)$, then $u(t, x) = \hat{u}(t, x)$.

Suppose in addition that (A.4) holds. Then:

(d) For any $t > 0$, $\partial u(t, \mathbf{R}^d) = \partial h(\mathbf{R}^d)$.

$$\int_{\mathbf{R}^d} (u(t, x) - h(x)) dx = t \cdot w(R, h, \mathbf{R}^d).$$

(e) Let $\bar{u} \in C([0, \infty) \times \mathbf{R}^d : \mathbf{R})$ be the solution to (1.2) with $u(0, \cdot) = \hat{h}$. If $u(s, \cdot) - \hat{u}(s, \cdot) \neq h - \hat{h}$ for some $s \in (0, \infty)$, then $\bar{u}(t, \cdot) - \hat{u}(t, \cdot) \neq 0$ for all $t \geq s$.

According to the above theorem, (a) any graph moves upward by R -curvature, (b) those points on any graph moving by R -curvature do not move as far as they stay in its cavities, (c) the height between any graph moving by R -curvature and its convex envelope is nonincreasing as it evolves, (d) any graph moving by R -curvature sweeps in time $t > 0$ a region with volume given by $t \cdot w(R, h, \mathbf{R}^d)$, and (e) for the motion of a graph by R -curvature, taking its convex envelope at time $t > 0$ and evolving up to time t starting with the convex envelope of the initial graph give different graphs in general, if the initial graph is not convex.

3. Motion by R -curvature and the viscosity solution.

In this section we discuss the relation between the motion by R -curvature and the viscosity solution to (PDE).

(A.5). $R \in C(\mathbf{R}^d : [0, \infty))$.

Theorem 3 Suppose that (A.1) and (A.5) hold. Then a continuous solution u to (1.2) with $u(0, \cdot) = h$ is a viscosity solution to (PDE).

Theorem 3 means that the motion by R -curvature is the viscosity solution to (PDE). To discuss under what condition the reverse is true, we discuss the uniqueness of the viscosity solution to (PDE).

(A.6). $R(x) \geq R(rx)$ for any $r \geq 1$ and $x \in \mathbf{R}^d$.

(A.7). $\inf_{x \neq o} h(x)/|x| > 0$.

(A.8). There exists a constant $C > 0$ such that $h(x+y)+h(x-y)-2h(x) \leq C$ for all $(x, y) \in \mathbf{R}^d \times U_1(o)$, where $U_1(o) \equiv \{y \in \mathbf{R}^d : |y| < 1\}$.

Theorem 4 *Suppose that (A.1) and (A.3)-(A.8) hold. Then there exists a unique continuous viscosity solution u to (PDE) with $u(0, \cdot) = h$ in the space of continuous functions v for which*

$$\sup\{|v(t, x) - h(x)| : (t, x) \in [0, T] \times \mathbf{R}^d\} < \infty \text{ for all } T > 0.$$

u is also a unique continuous solution to (1.2) with $u(0, \cdot) = h$.

We restrict our attention to Gauss curvature flow and give a finer result.

For $A \subset \mathbf{R}^d$ and $v : A \mapsto \mathbf{R}$, put

$$\text{epi}(v) = \{(x, y) : x \in A, y \geq v(x)\}.$$

For $r > 0$, put

$$h^r(x) = \inf\{y \in \mathbf{R} \mid U_r((x, y)) \subset \text{epi}(h)\} \quad (x \in \mathbf{R}^d).$$

Under the following condition, we give the comparison theorem for the continuous viscosity solution to (PDE).

(A.1)'. $R(y) = (1 + |y|^2)^{-(d+1)/2}$ and $h \in C(\mathbf{R}^d : \mathbf{R})$.

(A.2)'.

$$\liminf_{\theta \downarrow 1} \{ \liminf_{r \rightarrow \infty} [\liminf_{|x| \rightarrow \infty} (h(\theta x) - h^r(x))] \} > 0,$$

$$\lim_{\theta \downarrow 1} \{ \sup_{x \in \mathbf{R}^d} (h(x) - h(\theta x)) \} = 0.$$

Theorem 5 *Suppose that (A.1)'-(A.2)' hold. Then for any viscosity subsolution u and supersolution v , of (PDE) in the space $C([0, \infty) \times \mathbf{R}^d : \mathbf{R})$, such that $u(0, \cdot) \leq h \leq v(0, \cdot)$, $u \leq v$.*

Remark 3 *(A.2)' holds if there exists a convex function $h_0 : \mathbf{R}^d \mapsto \mathbf{R}$ such that $h_0(x) \rightarrow \infty$ as $|x| \rightarrow \infty$ and that*

$$\lim_{|x| \rightarrow \infty} [h(x) - h_0(x)] = 0.$$

In fact, the following holds:

$$\lim_{|x| \rightarrow \infty} [h(\theta x) - h^r(x)] = \infty \quad \text{for all } \theta > 1, r > 0,$$

$$\lim_{\theta \downarrow 1} \{ \sup_{x \in \mathbf{R}^d} [h(x) - h(\theta x)] \} = 0.$$

The following corollary is better than Theorem 4 in that we can consider the viscosity solution in the entire space $C(\mathbf{R}^d : \mathbf{R})$.

Corollary 1 *Suppose that (A.1)'-(A.2)' and (A.3)-(A.4) hold. Then there exists a unique continuous viscosity solution u to (PDE) with $u(0, \cdot) = h$. u is also a unique continuous solution to (1.2) with $u(0, \cdot) = h$.*

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G. Barles, S. Biton and O. Ley, Quelques résultats d'unicité pour l'équation du mouvement par courbure moyenne dans \mathbf{R}^N , preprint, Theorem 4.1,

where they studied a similar result to Theorem 5 for the mean curvature flow with a convex coercive initial function.